Logistic Regression

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Logistic Regression

Introduction

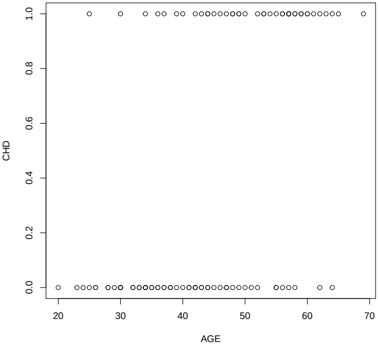
- *Logistic Regression* deals with the case where the dependent variable is binary, and the conditional distribution is binomial.
- Recall that, for a random variable Y having a binomial distribution with parameters n (the number of trials), and p (the probability of "success" , the mean of Y is np and the variance of Y is $np(1-p)$.
- Therefore, if the conditional distribution of Y given a predictor X is binomial, then the mean function and variance functions will be necessarily related.
- Moreover, since, for a given value of n, the mean of the conditional distribution is necessarily bounded by 0 and n, it follows that a linear function will generally fail to fit at large values of the predictor.
- So, special methods are called for.

Logistic Regression

Coronary Heart Disease

- As an example, consider some data relating age to the presence of coronary disease.
- The independent variable is the age of the subject, and the dependent variable is binary, reflecting the presence or absence of coronary heart disease.

```
> chd.data <- read.table(
+   "http://www.statpower.net/R312/chdage.txt", header=T)
> attach(chd.data)
> plot(AGE,CHD)
```



Logistic Regression

Coronary Heart Disease

- The general trend, that age is related to coronary heart disease, seems clear from the plot, but it is difficult to see the precise nature of the relationship.
- We can get a crude but somewhat more revealing picture of the relationship between the two variables by collecting the data in groups of ten observations and plotting mean age against the proportion of individuals with CHD.

Logistic Regression

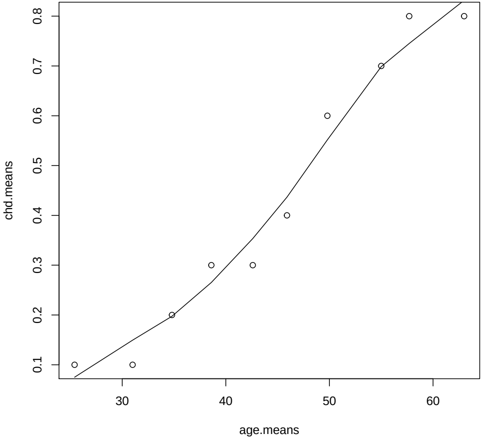
Coronary Heart Disease

```
> age.means <- rep(0,10)
> chd.means <- rep(0,10)
> for(i in 0:9)age.means[i+1]<-mean(
+   chd.data[(10*i+1):(10*i+10),2])
> age.means

 [1] 25.4 31.0 34.8 38.6 42.6 45.9 49.8 55.0 57.7 63.0

> for(i in 0:9)chd.means[i+1]<-mean(
+   chd.data[(10*i+1):(10*i+10),3])
> chd.means

 [1] 0.1 0.1 0.2 0.3 0.3 0.4 0.6 0.7 0.8 0.8
```

Logistic Regression

Coronary Heart Disease

```
> plot(age.means,chd.means)
> lines(lowess(age.means,chd.means,iter=1,f=2/3))
```



The Model

- For notational simplicity, suppose we have a single predictor, and define $p(x) = \Pr(Y = 1|X = x) = \mathrm{E}(Y|X = x)$.
- Suppose that, instead of the probability of heart disease, we consider the *odds* as a function of age.
- Odds range from zero to infinity, so the problem fitting a linear model to the upper asymptote can be eliminated.
- If we go one step further and consider the logarithm of the odds, we now have a dependent variable that ranges from $-\infty$ to $+\infty$.

The Model

- Suppose we try to fit a linear regression model to the log-odds variable.
- Our model would now be

$$\text{logit}(p(x)) = \log\left(\frac{p(x)}{1-p(x)}\right) = \beta_0 + \beta_1 x \tag{1}$$

If we can successfully fit this linear model, then we also have successfully fit a nonlinear model for $p(x)$, since the logit function is invertible, so after taking logit^{-1} of both sides, we obtain

$$p(x) = \text{logit}^{-1}(\beta_0 + \beta_1 x) \tag{2}$$

where

$$\text{logit}^{-1}(w) = \frac{\exp(w)}{1+\exp(w)} = \frac{1}{1+\exp(-w)} \tag{3}$$

The Model

- The above system generalizes to more than one predictor, i.e.,

$$p(\mathbf{x}) = \mathrm{E}(Y|X = \mathbf{x}) = \mathrm{logit}^{-1}(\boldsymbol{\beta}'\mathbf{x}) \tag{4}$$

The Model

- It turns out that the system we have just described is a special case of what is now termed a *generalized linear model*.
- In the context of generalized linear model theory, the logit function that "linearizes" the binomial proportions $p(\mathbf{x})$ is called a *link function*.
- In this module, we shall pursue logistic regression primarily from the practical standpoint of obtaining estimates and interpreting the results.
- Logistic regression is applied very widely in the medical and social sciences, and entire books on applied logistic regression are available.

Fitting with glm

- Fitting a logistic regression model in R is straightforward.
- You use the `glm` function and specify the binomial distribution family and the logit link function.

Fitting with glm

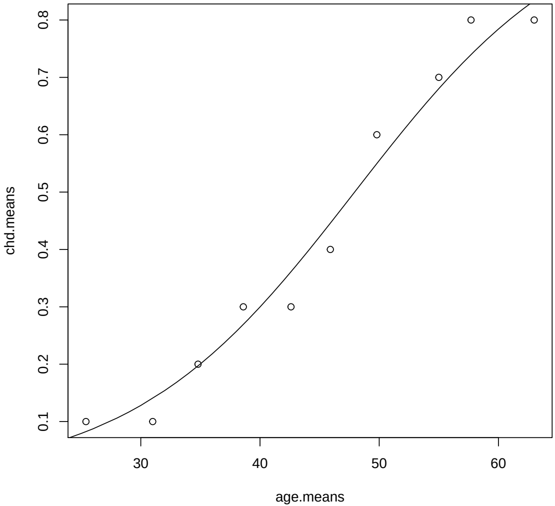
```
> fit.chd <- glm(CHD ~AGE, family=binomial(link="logit"))
> summary(fit.chd)


Call:
glm(formula = CHD ~ AGE, family = binomial(link = "logit"))

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.9407  -0.8538  -0.4735   0.8392   2.2518

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -5.12630    1.11205   -4.61 4.03e-06 ***
AGE          0.10695    0.02361    4.53 5.91e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 136.66  on 99  degrees of freedom
Residual deviance: 108.88  on 98  degrees of freedom
AIC: 112.88

Number of Fisher Scoring iterations: 4
```

Plotting Model Fit

- Remember that the coefficient estimates are for the transformed model. They provide a linear fit for $\text{logit}(p(x))$, not for $p(x)$. However, if we define an inverse logit function, we can transform our model back to the original metric.
- Below, we plot the mean `AGE` against the mean `CHD` for groups of 10 observations, then superimpose the logistic regression fit, transformed back into the probability metric.

```
> pdf("Scatterplot02.pdf")
> logit.inverse <- function(x){1/(1+exp(-x))}
> plot(age.means,chd.means)
> lines(AGE,logit.inverse(predict(fit.chd)))
```

Plotting Model Fit



Interpreting Model Coefficients

Binary Predictor

- Suppose there is a single predictor, and it is categorical (0,1). How can one interpret the coefficient β_1?
- Consider the *odds ratio*, the ratio of the odds when $x = 1$ to the odds when $x = 0$.
- According to our model, $\text{logit}(p(x)) = \exp(\beta_0 + \beta_1 x)$, so the log of the odds ratio is given by

$$\begin{aligned}
\log(OR) &= \log\left[\frac{p(1)/(1-p(1))}{p(0)/(1-p(0))}\right] \\
&= \log\left[p(1)/(1-p(1))\right] - \log\left[p(0)/(1-p(0))\right] \\
&= \text{logit}(p(1)) - \text{logit}(p(0)) \\
&= \beta_0 + \beta_1 \times 1 - (\beta_0 + \beta_1 \times 0) \\
&= \beta_1 \qquad (5)
\end{aligned}$$

Interpreting Model Coefficients

Binary Predictor

- Exponentiating both sides, we get

$$OR = \exp(\beta_1) \tag{6}$$

- Suppose that X represents the presence or absence of a medical treatment, and $\beta_1 = 2$. This means that the odds ratio is $\exp(2) = 7.389$. If the event is survival, this implies that the odds of surviving are 7.389 times as high when the treatment is present than when it is not.
- You can see why logistic regression is very popular in medical research, and why there is a tradition of working in the "odds metric."

Interpreting Model Coefficients

Continuous Predictor

- In our coronary heart disease data set, the predictor is continuous.
- Interpreting model coefficients when a predictor is continuous is more difficult.
- Recalling the form of the fitted function for $p(x)$, we see that it does not have a constant slope.
- By taking derivatives, we compute the slope as $\beta_1 p(x)(1 - p(x))$. Hence, the steepest slope is at $p(x) = 1/2$, at which $x = -\beta_0/\beta_1$, and the actual slope is $\beta_1/4$.
- In toxicology, this is called LD_{50}, because it is the dose at which the probability of death is $1/2$.

Interpreting Model Coefficients

Continuous Predictor

- So a rough "rule of thumb" is that when X is near the middle of its range, a unit change in X results in a change of $\beta_1/4$ units in $p(x)$.
- More precise calculations can be achieved with the aid of R and the logit^{-1} function.

Interpreting Model Coefficients

Continuous Predictor

Example (CHD vs. AGE)

- We saw that, in our CHD data, the estimated value of β_1 is 0.1069, and the estimated value of β_0 is -5.1263.
- This suggests that, around the age of 45, an increase of 1 year in *AGE* corresponds roughly to an increase of 0.0267 in the probability of coronary heart disease.
- Let's do the calculations by hand, using R.

```
> beta.1 <- coefficients(fit.chd)[2]
> beta.0 <- coefficients(fit.chd)[1]
> predict.45 <- logit.inverse(beta.0 + beta.1 * 45)
> predict.46 <- logit.inverse(beta.0 + beta.1 * 46)
>   change <- predict.46 - predict.45
> results <- data.frame(t(as.numeric(c(predict.45,
+   predict.46,change, beta.1/4))))
> colnames(results) <- c("predict.45","predict.46",
+   "change",".25*beta.1")
> results
  predict.45 predict.46     change .25*beta.1
1   0.422195  0.4484776 0.02628253 0.02673629
```

Interpreting Model Coefficients

Continuous Predictor

- The numbers demonstrate that, in the "linear zone" near the center of the plot, the rule of thumb works quite well.
- The rule implies that for every increase of 4 units in *AGE*, there will be roughly a β_1 increase in the probability of coronary heart disease.
- We can simplify the calculations on the preceding slide by using the `predict` function on the fit object.

Interpreting Model Coefficients

Continuous Predictor

Example (CHD vs. AGE)

- Suppose we wish to obtain predicted probabilities for ages 45 through 50.
- We set up a data frame with the new `AGE` data. Note that you must use the exact same name as the predictor variable in the data frame you analyzed.

```
> my.data <- data.frame(45:50)
> colnames(my.data) <- c("AGE")
> rownames(my.data) <- as.character(my.data$AGE)
```

- Using the `predict` function is straightforward.
- However, to obtain the values in the correct (probability) metric, we must remember to use the `type = "response"` option!

```
> predict(fit.chd,newdata = my.data,type="response")
       45        46        47        48        49
0.4221950 0.4484776 0.4750511 0.5017666 0.5284721
       50
0.5550155
```

Assessing Model Fit in Logistic Regression

Deviance

- In multiple linear regression, the residual sum of squares provides the basis for tests for comparing mean functions.
- In logistic regression, the residual sum of squares is replaced by the *deviance*, which is often called G^2. Suppose there are k data groupings based on $n_i, i = 1, \ldots, k$ binomial observations. The deviance is defined for logistic regression to be

  $$G^2 = 2 \sum_{i=1}^{k} \left[y_i \log \left(\frac{y_i}{\hat{y}_i} \right) + (n_i - y_i) \log \left(\frac{n_i - y_i}{n_i - \hat{y}_i} \right) \right] \tag{7}$$

  where $\hat{y}_i = n_i \hat{p}(\mathbf{x}_i)$ are the fitted numbers of successes in n_i trials in the ith grouping.
- The degrees of freedom associated with the analysis is the number of groupings n used in the calculation minus the number of free parameters in β that were estimated.

Comparing Models

- Comparing models in logistic regression is similar to regular linear regression.
- For two nested models, the difference in deviances is treated as a chi-square with degrees of freedom equal to the difference in the degrees of freedom for the two models.

Test of Model Fit

- When the number of trials $n_i > 1$, the deviance G^2 can be used to provide a goodness-of-fit test for a logistic regression model.
- The test compares the null hypothesis that the mean function used is adequate versus the alternative that a separate parameter needs to be fit for each value of i (this latter case is called the *saturated model*).
- When all the n_i are large enough, G^2 can be compared with the χ^2_{n-p} distribution to get an approximate p-value.

Test of Model Fit

- An alternative statistic is the Pearson X^2

$$X^2 = \sum_{i=1}^{k} \left[(y_i - \hat{y}_i)^2 \left(\frac{1}{\hat{y}_i} + \frac{1}{n_i - \hat{y}_i} \right) \right] = \sum_{i=1}^{k} \frac{n_i (y_i/n_i - \hat{\theta}(\mathbf{x}_i))^2}{\hat{\theta}(\mathbf{x}_i)(1 - \hat{\theta}(\mathbf{x}_i))} \tag{8}$$

- According to ALR, X^2 and G^2 have the same large-sample distribution and often give the same inferences. But in small samples, there may be differences, and sometimes X^2 may be preferred for testing goodness-of-fit.

Logistic Regression with Several Predictors

The Titanic Disaster

- As an example of logistic predictors, Weisberg presents data from the famous Titanic disaster. (Frank Harrell presents a much more detailed analysis of the Titanic in his superb book *Regression Modeling Strategies*).
- Of 2201 known passengers and crew, only 711 are reported to have survived.
- The data in the file `titanic.txt` from Dawson (1995) classify the people on board the ship according to their *Sex* as Male or Female, *Age*, either child or adult, and *Class*, either first, second, third, or crew.
- Not all combinations of the three factors occur in the data, since no children were members of the crew. For each age/sex/class combination, the number of people M and the number surviving *Surv* are also reported.
- The data are shown in Table 12.5.

Logistic Regression with Several Predictors

The Titanic Disaster

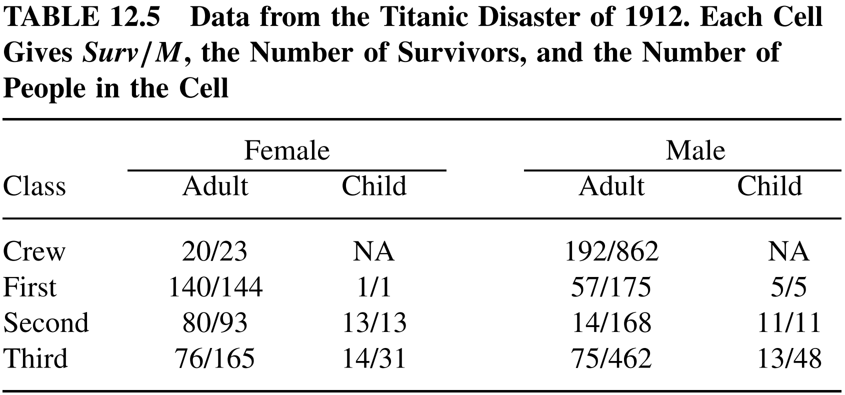


TABLE 12.5 Data from the Titanic Disaster of 1912. Each Cell Gives *Surv*/*M*, the Number of Survivors, and the Number of People in the Cell

	Female		Male	
Class	Adult	Child	Adult	Child
Crew	20/23	NA	192/862	NA
First	140/144	1/1	57/175	5/5
Second	80/93	13/13	14/168	11/11
Third	76/165	14/31	75/462	13/48

Logistic Regression with Several Predictors

The Titanic Disaster

- ALR fits a sequence of 5 models to these data.
- Since almost all the m_i exceed 1, we can use either G^2 or X^2 as a goodness-of-fit test for these models.
- The first two mean functions, the main effects only model, and the main effects plus the $Class \times Sex$ interaction, clearly do not fit the data because the values of G^2 and X^2 are both much larger then their df, and the corresponding *p*-values from the χ^2 distribution are 0 to several decimal places.
- The third model, which adds the $Class \times Age$ interaction, has both G^2 and X^2 smaller than its df, with *p*-values of about 0.64, so this mean function seems to match the data well.
- Adding more terms can only reduce the value of G^2 and X^2, and adding the third interaction decreases these statistics to 0 to the accuracy shown.
- Adding the three-factor interaction fits one parameter for each cell, effectively estimating the probability of survival by the observed probability of survival in each cell. This will give an exact fit to the data.

Logistic Regression with Several Predictors

The Titanic Disaster

```
> library(alr3)

Loading required package:  car

> library(xtable)
> mysummary <- function(m){c(df=m$df.residual,G2=m$deviance,
+                             X2=sum(residuals(m,type="pearson")^2) )}
> m1 <- glm(cbind(Surv,N-Surv)~Class+Age+Sex, data=titanic, family=binomial())
> m2 <- update(m1,~.+Class:Sex)
> m3 <- update(m2,~.+Class:Age)
> m4 <- update(m3,~.+Age:Sex)
> m5 <- update(m4,~Class:Age:Sex)
> ans <- mysummary(m1)
> ans <- rbind(ans,mysummary(m2))
> ans <- rbind(ans,mysummary(m3))
> ans <- rbind(ans,mysummary(m4))
> ans <- rbind(ans,mysummary(m5))
>  row.names(ans) <- c( "Main effects only",
+  "Main Effects + Class:Sex",
+  "Main Effects + Class:Sex +  Class:Age",
+  "Main Effects + All 2 Factor Interactions",
+  "Main Effects + All 2 and 3 Factor Interactions")
```

Logistic Regression with Several Predictors

The Titanic Disaster

```
> options(scipen=1,digits=3)
> summary(m3)


Call:
glm(formula = cbind(Surv, N - Surv) ~ Class + Age + Sex + Class:Sex +
    Class:Age, family = binomial(), data = titanic)

Deviance Residuals:
      1        2        3        4        5        6
 0.0000   0.0000   0.0000   0.0001   0.0000   0.0000
      7        8        9       10       11       12
 0.0000   0.0001   0.0000   0.0000  -0.8745   0.8265
     13       14
 0.3806  -0.3043

Coefficients: (1 not defined because of singularities)
                      Estimate Std. Error z value
(Intercept)              1.897      0.619    3.06
ClassFirst               1.658      0.800    2.07
ClassSecond             -0.080      0.688   -0.12
ClassThird              -2.115      0.637   -3.32
AgeChild                 0.338      0.269    1.26
SexMale                 -3.147      0.625   -5.04
ClassFirst:SexMale      -1.136      0.821   -1.38
ClassSecond:SexMale     -1.068      0.747   -1.43
ClassThird:SexMale       1.762      0.652    2.70
ClassFirst:AgeChild     22.424  16495.727    0.00
ClassSecond:AgeChild    24.422  13007.888    0.00
ClassThird:AgeChild         NA         NA      NA
                     Pr(>|z|)
(Intercept)            0.0022 **
ClassFirst             0.0383 *
ClassSecond            0.9073
ClassThird             0.0009 ***
AgeChild               0.2094
SexMale               4.7e-07 ***
ClassFirst:SexMale     0.1662
ClassSecond:SexMale    0.1525
ClassThird:SexMale     0.0069 **
ClassFirst:AgeChild    0.9989
ClassSecond:AgeChild   0.9985
ClassThird:AgeChild        NA
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 671.9622  on 13  degrees of freedom
Residual deviance:   1.6854  on  3  degrees of freedom
AIC: 70.31

Number of Fisher Scoring iterations: 21
```

Logistic Regression with Several Predictors

The Titanic Disaster

```
> xtable(ans)
```

	df	G2	X2
Main effects only	8.00	112.57	103.83
Main Effects + Class:Sex	5.00	45.90	42.77
Main Effects + Class:Sex + Class:Age	3.00	1.69	1.72
Main Effects + All 2 Factor Interactions	2.00	0.00	0.00
Main Effects + All 2 and 3 Factor Interactions	0.00	0.00	0.00

Generalized Linear Models

- Both the multiple linear regression model discussed earlier in this book and the logistic regression model discussed in this chapter are particular instances of a *generalized linear model*.
- Generalized linear models all share three basic characteristics:

Generalized Linear Models

1. The distribution of the response Y, given a set of terms X, is distributed according to an exponential family distribution. The important members of this class include the normal and binomial distributions we have already encountered, as well as the Poisson and gamma distributions.
2. The response Y depends on the terms X only through the linear combination $\beta'\mathbf{X}$.
3. The mean $\mathrm{E}(Y|X = \mathbf{x}) = \mathbf{m}(\beta'\mathbf{x})$ for some kernel mean function $\mathbf{m}$. For the multiple linear regression model, $\mathbf{m}$ is the identity function, and for logistic regression, it is the logistic function. There is considerable flexibility in selecting the kernel mean function. Most presentations of generalized linear models discuss the link function, which technically is defined as the *inverse* of $\mathbf{m}$ rather than $\mathbf{m}$ itself.

Classification Via Logistic Regression

- In some previous lectures, we discussed *discriminant analysis* and its use as a method of *classification*.
- Since binary logistic regression provides a predicted probability of the two binary outcomes, one may classify observations using logistic regression, as we demonstrate in the following example.
- We download some data representing measurements of human skulls dating back to ancient Egypt.
- This subset of the original data set may be downloaded from the website.

```
> Egypt <- read.csv(
+   "http://www.statpower.net/R312/Egypt.csv"
+   )
```

Classification Via Logistic Regression

Egyptian Skull Data

- Below is the key information on the data

```
> names(Egypt)

[1] "Group" "mb"    "bh"    "bl"    "nh"

> #Group 1 = circa 4000 BC
> #Group 2 = circa 3300 BC
> #mb: maximum breadth of the skull.
> #bh: basibregmatic height of the skull.
> #bl: basialiveolar length of the skull.
> #nh: nasal heights of the skull.
```

Classification Via Logistic Regression

Egyptian Skull Data

- We predict the probabilities for membership in the 4000 B.C. or 3300 B.C. epochs from the skull measurements.

```
> Egypt$Group <- Egypt$Group-1  #convert to binary variable
> fit <- glm(Group ~ ., data = Egypt, family=binomial(link="logit"))
> summary(fit)


Call:
glm(formula = Group ~ ., family = binomial(link = "logit"), data = Egypt)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.4402  -1.1406  -0.0959   1.1515   1.4905

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  1.83842   10.97726    0.17     0.87
mb           0.05763    0.05791    1.00     0.32
bh          -0.04345    0.06064   -0.72     0.47
bl          -0.00904    0.05210   -0.17     0.86
nh          -0.05468    0.10147   -0.54     0.59

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 83.178  on 59  degrees of freedom
Residual deviance: 81.492  on 55  degrees of freedom
AIC: 91.49

Number of Fisher Scoring iterations: 4
```

Classification Via Logistic Regression

Egyptian Skull Data

- None of the coefficients is statistically significant, suggesting that this logistic regression will not do a very good job of classifying the skulls.
- We can produce a classification table in a very straightforward manner from the fit object.
- As expected, the performance is not much better than chance.

```
> Class <- predict(fit,type="response")>.5
> table(Egypt$Group,Class)

   Class
    FALSE TRUE
  0    18   12
  1    14   16
```

Classification Via Logistic Regression

Egyptian Skull Data

- Let's try discriminant analysis on the data and see what happens.

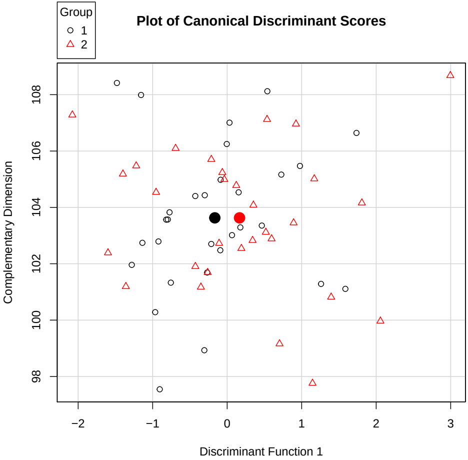
```
> source("http://www.statpower.net/Content/312/R Stuff/Steiger R Libra
> source("http://www.statpower.net/Content/312/R Stuff/ClassifyCode.r"
> x <- as.matrix(Egypt[,2:5])
> Group <- as.matrix(Egypt[,1]) + 1
> out <- Classify(x,Group)
> out$Classification.Table

     Classified
Group  1  2
    1 19 11
    2 14 16
```

- The plot of the scores on the next slide shows that there is not much separation between the groups in discriminant space.

Classification Via Logistic Regression

Egyptian Skull Data

```
> D <- Make.D(Group)
> H <- Make.H(Group)
> Plot.Discriminant.Scores(x,D,H,Group)
```



Classification Via Logistic Regression

Egyptian Skull Data

- The canonical table confirms the lack of statistical significance.

```
> print(Canonical.Table(x,D,H))

     Fcn  Eigen Prop CanCorr Lambda F-Stat df1 df2
[1,]   1 0.0285    1   0.166  0.972  0.391   4  55
      prob
[1,] 0.814
```

- In their sign and relative size, the standardized discriminant weights closely match the pattern of the logistic regression weights.

```
> print(Standardized.Discriminant.Weights(x,D,H))

    mb     bh     bl     nh
 0.832 -0.577 -0.138 -0.457
```

Classification Via Logistic Regression

Egyptian Skull Data

- The bottom line is that in quite a few situations, logistic regression will produce results similar to linear discriminant analysis.
- Logistic regression, however, makes fewer statistical assumptions. It does not require continuous predictors, and it therefore naturally does not require multivariate normality.

Multinomial Logistic Regression

Football Data

- The binary logistic regression model involving two outcomes generalizes to the multinomial logistic regression model.
- A rudimentary procedure to fit multinomial regression models is available in the `nnet` library.
- Here we quickly demonstrate classifying the football data.

Multinomial Logistic Regression

Football Data

- Here is the code.

```
> fb.data <- read.table(
+   "http://www.statpower.net/R312/football.txt",header=T,sep=",")
> names(fb.data)

[1] "GROUP"  "WDIM"   "CIRCUM" "FBEYE"  "EYEHD"
[6] "EARHD"  "JAW"

> library(nnet)
> mod <- multinom(GROUP ~.,fb.data)

# weights:  24 (14 variable)
initial  value 98.875106
iter  10 value 53.052168
iter  20 value 51.037137
iter  30 value 50.193419
iter  40 value 50.102582
iter  50 value 50.086496
final  value 50.072216
converged

> table(fb.data$GROUP,predict(mod))


     1  2  3
  1 27  2  1
  2  1 20  9
  3  2  8 20
```

Multinomial Logistic Regression

Football Data

```
> summary(mod)

Call:
multinom(formula = GROUP ~ ., data = fb.data)

Coefficients:
  (Intercept) WDIM CIRCUM FBEYE EYEHD EARHD   JAW
2        26.4 3.91 -0.259   1.6 -2.31 -1.89 -4.13
3        21.6 5.21 -0.435   1.7 -1.58 -2.05 -5.19

Std. Errors:
  (Intercept) WDIM CIRCUM FBEYE EYEHD EARHD  JAW
2        5.44 1.97  0.519  1.31 0.650 0.869 1.98
3        5.87 1.98  0.489  1.28 0.604 0.852 1.98

Residual Deviance: 100
AIC: 128
```